Closed Form Asymptotic Expression of a Random-Access Interference Measure

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***Abstract*—A model describing the cumulative effect of the independent access of K users to a shared resource, which is formed by N elements, is proposed, based on which an integer interference measure ζ is defined. While traditional cases can be reconducted to reference well-known results, for which ζ is either Gaussian or Poissonian, the proposed model provides a more general framework that offers the tool for understanding the nature of ζ. In particular, an asymptotic closed form expression ($K \to \infty$, $N \to \infty$, $K/N \to \beta \in (0, \infty)$) for ζ distribution is provided for systems presenting constructive versus destructive interference, and as such is applicable to characterizing statistical properties of interference in a wide range of random multiple access channels.**

***Index Terms*—Interference, multiple access channels, large-system limit analysis.**

I. Introduction and System Model

When many users access a common resource independently, they may interfere with each other. A resource can be viewed, in general, as a set of elements that are used to transmit information. For example, at the physical layer, the resource is the set of degrees of freedom that carry the information-bearing signal: a system using bandwidth W for time T with N_t antennas can access WTN_t degrees of freedom belonging to time, frequency and space domains; at the medium access layer, the resource is usually time supporting either continuous or slotted packet transmission.

In the proposed model, the resource is a discrete set of N elements[1] $[1 : N]$, i.e., N *slots*. This description holds when the resource is discrete, or can be aptly discretized. Resource is shared independently by K users: user k chooses a subset $\mathcal{L}_k$ of L (irrespective of k) slots, ignoring the other users choice, and assigns a label $S^\star_{nk}$ to each accessed slot $n \in \mathcal{L}_k$.

Fig. 1 illustrates the abstract setting, where a resource is made up of N slots and K users access randomly to a subset of slots, which is shown for $L = 1$.

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[1]In this paper, two common notations in combinatorics are used: the set $\{m, m+1, \ldots, m+n-1\}$ is denoted by $[m : m+n-1]$, and, when $m = 1$, it is simply written as $[n]$.

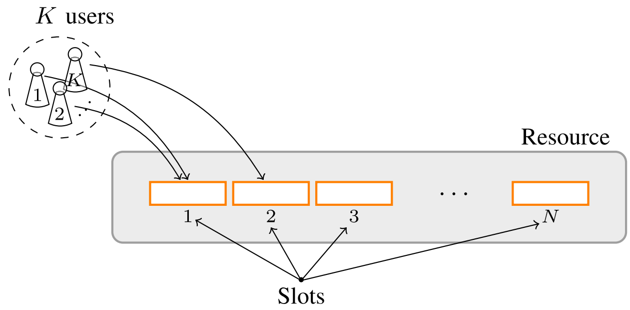

Fig. 1. Abstract setting: a resource composed on N parts (*slots*) is randomly accessed by K users. Each user assigns to the accessed slot a numerical value (*label*), randomly and independently from other users.

Let $S_{nk} = S^\star_{nk}$ for $n \in \mathcal{L}_k$, and $S_{nk} = 0$ otherwise, that is, S_{nk} is zero for the non-accessed slots of user k, while it is equal to the assigned labels for the accessed slots. Therefore, $S^\star_{nk} \in \mathcal{S}^\star$, where $\mathcal{S}^\star$ is the set of possible values that the label may assume, and $S_{nk} \in \mathcal{S}$, where $\mathcal{S}$ and $\mathcal{S}^\star$ may or may not be equal.

Let Z_{nk} be the sum of labels assigned to slot n by all users but user k, that is:

$$Z_{nk} = \sum_{\substack{i=1 \\ i \neq k}}^{K} S_{ni}. \tag{1}$$

By specifying $\mathcal{S}^\star$ and L, the proposed model encompasses the problem of statistically describing multiple access interference for communication systems in which interference has a quantal nature, or can be reconducted to the model of eq. (1). Let present four examples: the first three address well-known problems, and are intended to clarify concepts and notations defined above, while the last addresses a novel problem, that is solved thanks to results proved in this paper.

Example 1: This example may refer to throughput of Slotted Aloha at the medium access layer [1]. Assume that resource is time, that is slotted in N equal parts, and that K users, each willing to transmit a single packet, independently choose one of the N slots. In this setting, $L = 1$ and $S^\star_{nk}$ can be interpreted as the binary variable indicating the presence ($S^\star_{nk} = 1$) or absence ($S^\star_{nk} = 0$) of packet of user k within slot n. The goal is to find the number of colliding packets with user k packet, i.e., the number of packets in the slot n_k selected by user k. Given n_k, it results $S^\star_{nk} = \delta_{n,n_k}$. Moreover, $\mathcal{S}^\star = \{1\}$ and $\mathcal{S} = \{0, 1\}$. Z_{nk} in eq. (1) indicates the number of packets in slot n that are transmitted by all users but user k, and, therefore, the goal is to find $Z_{n_k k}$. In the large-system limit, that is, as $N \to \infty$, $K \to \infty$, and $(K/N) \to \beta \in (0, \infty)$, both Z_{nk} and $Z_{n_k k}$ are distributed according to a Poisson distribution with mean β.

Example 2: This example may describe demodulation of direct-sequence spread-spectrum (DSSS) signals with a single-user matched filter bank. Assume that resource is time, that is slotted in N chips of duration Δ, and supports transmission of synchronous random DSSS signals [6]. Let the received signal be

$$y(t) = \sum_{k=1}^{K} b_k s_k(t) + n(t), \tag{2}$$

where K is the number of users, $\{s_k(t)\}_{k=1}^K$ is the set of unit energy transmitted spreading waveforms, $\{b_k\}_{k=1}^K$ is the set of transmitted antipodal symbols, and $n(t)$ is a white Gaussian noise process with power spectrum σ_N^2. Direct-sequence implies $s_k(t) = \sum_{n=0}^{N-1} s_k[n]\psi(t - n\Delta)$, where $\{s_k[n] : k = 1, \ldots, K; n = 0, \ldots, N-1\}$ are i.i.d. r.v.s assuming values in $\{-(1/\sqrt{N}), (1/\sqrt{N})\}$ with equal probability (see e.g. [6]). In this scenario, $L = N$, $\mathcal{S}^\star = \{-(1/\sqrt{N}), (1/\sqrt{N})\}$, $\mathcal{S}^\star = \mathcal{S}$, $S_{nk} = b_k s_k[n]$, and Z_{nk} is the signal interfering with that of user k, within chip n, at the output of a chip-matched filter. The goal is to find the distribution of Z_{nk}, that can be used, for example, in order to find the capacity of the system. In the large-system limit, Z_{nk} is Gaussian distributed with zero mean and unit variance. Generally, a Lindeberg condition suggests that if a fixed fraction, however small, of degrees of freedom is "uniformly" used, then Z_{nk} is Gaussian distributed; otherwise, if few degrees of freedom are used in the large-system limit, e.g. a finite number, then the asymptotic distribution may be not Gaussian, as is usually the case.

Example 3: This example may describe demodulation of binary PPM time-hopping spread-spectrum (THSS) signals with a single-user matched filter bank, where interference is at physical rather than medium access layer compared to Example 1, and can only be constructive. Assume that resource is time, that is divided in $N/2$ chips of duration Δ, and supports transmission of synchronous binary PPM THSS signals (see e.g. [5]). Let the received signal be

$$y(t) = \sum_{k=1}^{K} \psi(t - c_k\Delta - \epsilon b_k) + n(t), \tag{3}$$

where K is the number of users, c_k is uniformly distributed over $[0 : (N/2) - 1]$ assuming $N/2$ an integer, $\{b_k\}_{k=1}^K$ is the set of binary transmitted symbols, and, for the sake of simplicity, $\psi(t)$ is a zero-excess bandwidth waveform with band $[-W/2, W/2]$, and $1/W = \Delta/2 = \epsilon$. In this model, there are N slots of duration $\Delta/2$, $\mathcal{S}^\star = \{1\}$, $\mathcal{S} = \{0, 1\}$, and Z_{nk} may be regarded as the interference of the output of a filter slot-matched to slot n. In the large-system limit, Z_{nk} is distributed according to a Poisson distribution with mean β, as in the first example.

Example 4: Example 4 is similar to Example 2, except for the random spreading sequences that now belong to the time-hopping family (see e.g. [3], [4]), i.e., for any fixed k, $s_k[n] \in \{-1, 1\}$, with equal probability, for only one chip $n_k \in [0 : N-1]$. In this case, Z_{nk} is the interference, to which contribute both constructive and distructive terms $b_i s_i[n]$ for $i \neq k$, at the output of a filter chip-matched to chip n, $\mathcal{S}^\star = \{-1, 1\}$, and $\mathcal{S} = \{-1, 0, 1\}$. Moreover, $Z_{n_k k}$ is the interference at the output of the single-user matched filter of user k. The distribution of $Z_{n_k k}$ is unknown, and can be found thanks to the result presented in this paper.

As hinted by Example 4, this paper finds, in the large-system limit, the closed form distribution of:

$$\zeta_k \triangleq Z_{n_k k} = \sum_{\substack{i=1 \\ i \neq k}}^{K} S_{n_k i}. \tag{4}$$

The paper is organized as follows: in Section II the main result is presented and proved; essential analytic combinatorics are recapped in the Appendix. Conclusions are drawn in Section III.

II. Main Result

Theorem 1: Let N be the number of slots of a resource, that is shared by K users. The generic user $k \in [1 : K]$ selects one slot only $n_k \in [1 : N]$, and assigns to this slot a label $S_{n_k k}^\star$ that is a r.v. taking value in $\{-1, 1\}$ with equal probability. Then, ζ_k, as defined in eq. (4), is distributed in the large-system limit, that is, for $N \to \infty$, $K \to \infty$, and $K/N \to \beta \in (0, \infty)$, as:

$$P_\zeta = \sum_{m \in \mathbb{Z}} e^{-\beta} I_m(\beta) \delta_m, \tag{5}$$

irrespective of k, where $I_m(\beta)$ is the modified Bessel function of the first kind.

Proof: We provide two proofs. The first proof is probabilistic: S_{nk} is regarded as a r.v. and the pdf of ζ is derived straightforwardly via algebraic manipulations. The second proof is based on results of analytic combinatorics: the probability $\mathbb{P}(\zeta = z)$ is derived by considering all the ways, and the associated probability, a particular value z of ζ can be obtained; the law of large numbers guarantees that the so obtained result holds with probability one in the large-system limit.

First Proof: For fixed k, ζ_k is the sum of $(K - 1)$ i.i.d. random variables $\{S_{n_k i}\}_{i=1, i \neq k}^K$, each of which is distributed according to:

$$P_S = \frac{1}{2N}\delta_{-1} + \left(1 - \frac{1}{N}\right)\delta_0 + \frac{1}{2N}\delta_1. \tag{6}$$

Denoting by $b(n, p; r) = \binom{n}{r} p^r (1-p)^{n-r}$, one has:

$$P_{\zeta_k} = \sum_{i=0}^{K-1} b(K-1, 1/N; i) \frac{1}{2^i} \sum_{\ell=0}^{i} \binom{i}{\ell} \delta_{2\ell - i}.$$

In order to obtain the asymptotic pdf, rewrite the previous relation as follows:

$$P_{\zeta_k} = \sum_{|m| < K} \left\{ \sum_{j=0}^{J_m} c_{|m|+2j, j} \right\} \delta_m, \quad J_m = \frac{K - 1 - |m|}{2},$$

where this time the contribution to the amplitude of each Dirac mass is isolated in the term in parentheses, that is defined as $c_{i\ell} = b(K-1, 1/N; i)(1/2^i)\binom{i}{\ell}$. In the large-system limit, the Binomial distribution tends to a Poisson distribution with mean β, $b(\beta N - 1, 1/N; i) \to e^{-\beta}(\beta^i / i!)$, and the term in parentheses reduces to $e^{-\beta} I_{|m|}(\beta)$, being:

$$I_m(\beta) = \sum_{i=0}^{\infty} \frac{1}{i!} \cdot \frac{1}{(m+i)!} \left(\frac{\beta}{2}\right)^{2i+m}.$$

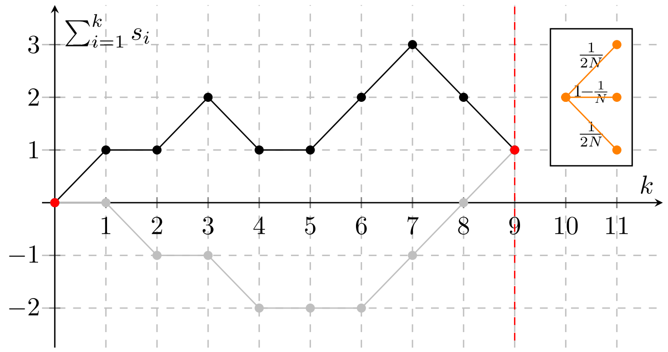

Fig. 2. Two simple lattice walks are shown. In the two cases, $K = 10$, and therefore the walk length is 9, and steps are $\mathcal{S} = \{-1, 0, 1\}$, as shown in the box at the north-east corner. Both walks start from (0,0) and end at (9,1): the darker corresponds to the sequence $(1, 0, 1, -1, 0, 1, 1, -1, -1)$, while the lighter corresponds to $(0, -1, 0, -1, 0, 0, 1, 1, 1)$.

Finally, P_{ζ_k} becomes:

$$P_\zeta = \sum_{m\in\mathbb{Z}} e^{-\beta} I_{|m|}(\beta)\delta_m,$$

irrespective of k. The theorem follows since $I_{|m|}(\beta) = I_m(\beta)$ when $m \in \mathbb{Z}$.

Second Proof: As expressed by (4), ζ_k is a sum of the form:

$$\zeta = \sum_{i=1}^{K-1} s_i, \quad s_i \in \{-1, 0, 1\},$$

where subscript k is discarded. In order to find $\mathbb{P}(\zeta = h)$, the number of ways h can be obtained as sum of elements of a sequence $(s_1, \ldots, s_{K-1})$ is counted, and let the probability each that sequence appears be:

$$\Pr(\zeta = h) = \sum_{\substack{s_k\in\mathcal{S}, k\in[K-1] \\ \sum_{k=1}^{K-1} s_k = h}} P(s_1, \ldots, s_{K-1}). \tag{7}$$

Sequence $(s_1, \ldots, s_{K-1})$ can be regarded as an unconstrained, simple walk of length $K - 1$ in the lattice $\mathbb{Z} \times \mathbb{Z}$ (refer to the Appendix for definitions and theorems of analytic combinatorics that are used in this paper). Fig. 2 shows two such walks with $|\mathcal{S}| = 3$ possible steps, $\mathcal{S} = \{-1, 0, 1\}$, where notation for simple walks is adopted. Associated with these steps are weights $1/(2N)$ and $(1 - (1/N))$ (see box at north-east corner of Fig. 2) such that the characteristic polynomial of $\mathcal{S}$ is:

$$P(y) = \frac{1}{2N}\frac{1}{y} + \left(1 - \frac{1}{N}\right) + \frac{1}{2N} y = 1 - \frac{1}{N}\left(1 - \frac{y}{2} - \frac{1}{2y}\right).$$

Thanks to weights, the probability a particular point in $\mathbb{Z} \times \mathbb{Z}$ is reached can be computed. In order to find $\mathbb{P}(\zeta = h)$, summation in eq. (7) is over walks starting from (0,0) and ending at $(K - 1, h)$, and the probability within the sum is that associated to each walk, that is the product of probabilities associated to steps composing the walk. The generating function of these walks is:

$$W(x, y) = \frac{1}{1 - xP(y)} = \sum_{k\ge 0} P(y)^k x^k,$$

the coefficient $[x^{K-1}y^h]W(x, y)$ giving the probability to reach $(K - 1, h)$:

$$\begin{aligned}[x^{K-1}y^h]W(x, y) &= [y^h]P(y)^{K-1} \\ &= [y^h]\left[1 - \frac{1}{N}\left(1 - \frac{y}{2} - \frac{1}{2y}\right)\right]^{K-1}.\end{aligned}$$

In the large-system limit, the quantity in brackets converges to:

$$[x^{K-1}]W(x, y) \longrightarrow e^{\beta\alpha(y)} \triangleq \bar{P}(y),$$

with $\alpha(y) = 1 - (y/2) - (1/2y) = -(y - 1)^2/2y$. In order to find $[y^h]\bar{P}(y)$ and therefore $[x^{K-1}y^h]W(x, y)$, Cauchy's integral formula can be used as follows:

$$\begin{aligned}[y^h]\bar{P}(y) &= \frac{1}{2\pi j}\oint_\gamma \frac{\bar{P}(y)}{y^h}\cdot\frac{dy}{y} = \frac{1}{2\pi j}\oint_{|y|=1} \frac{\bar{P}(y)}{y^h}\cdot\frac{dy}{y} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \bar{P}(e^{j\theta})e^{-jh\theta}d\theta \\ &= e^{-\beta}\int_0^{2\pi} e^{\beta\cos\theta}e^{-jh\theta}d\theta = e^{-\beta}I_h(\beta).\end{aligned}$$

Therefore, ζ assumes the integer value h with probability $e^{-\beta}I_h(\beta)$, hence the theorem. □

Fig. 3 shows simulations (filled circles at integer values) versus theoretical envelope $e^{-\beta}I_{|z|}(\beta)$, $z \in \mathbb{R}$ (red solid line) of P_ζ, for $\beta = 1/2$ (Fig. 3(a)) and $\beta = 2$ (Fig. 3(b)). Simulated values are drawn from Monte-Carlo simulations of 10^5 finite dimensional systems with $N = 100$. A Gaussian r.v. with same mean and variance (black dashed line) is reported for reference. As hinted by figures, the envelope of the distribution is increasingly Gaussian as β increases. In particular, odd moments of ζ are null, while the two first even moments are $\mathbb{E}[\zeta^2] = \beta$ and $\mathbb{E}[\zeta^4] = \beta(1 + 3\beta)$, hence the kurtosis is $\kappa = 3 + 1/\beta$. Since $\beta > 0$, ζ is always leptokurtic, and $\kappa \to 3$ as $\beta \to \infty$, suggesting that a Gaussian approximation may hold for $\beta \gg 1$.

III. Conclusion

In this paper, a model describing systems where users access a resource independently was proposed. Each user assigns labels to accessed slots: the label is a numerical value with sign, i.e., accounting for polarity. Based on this model, an interference measure called ζ_k for the generic user k that considers the cumulative value of other users labels in terms of their sum was considered. In particular, the case where each user accesses one slot only and assigns a label -1 or 1 with equal probability to the accessed slot was addressed. A closed form expression of the distribution of this cumulative value was found in the large-system limit: it was shown that, if the cardinality of the population of users is a fraction β of the number of available slots, then the distribution converges to a novel expression that is in general far from Gaussian, and may be approximated by a Gaussian distribution for $\beta \gg 1$. Two proofs, one probabilistic and the other based on analytic combinatorics, were provided. In particular, the second proof presents a potentially fruitful framework that can be used to derive further generalizations.

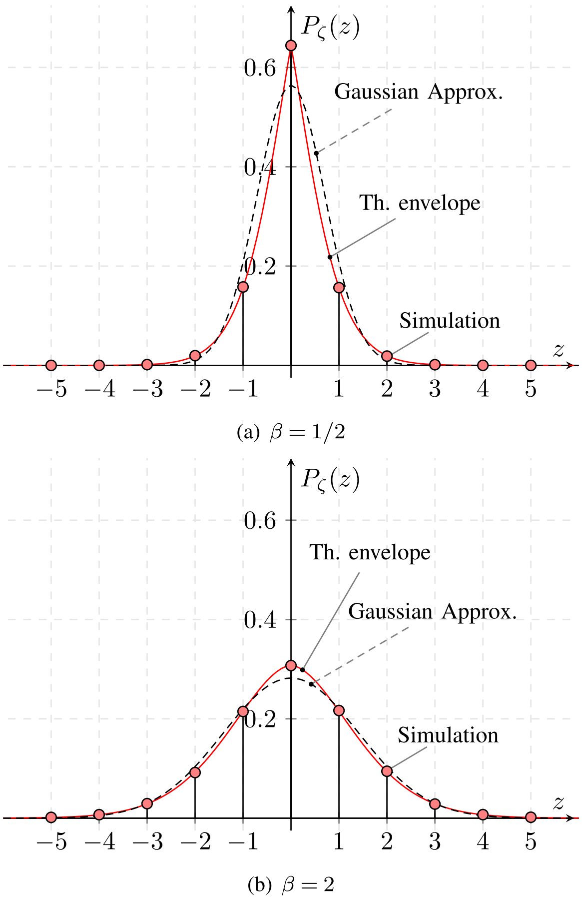

Fig. 3. Theoretical envelope (red dashed line) versus simulated histogram (filled circles) of (a) ζ for $\beta = 1/2$ and (b) $\beta = 2$. A Gaussian pdf with same mean and variance is shown for reference.

APPENDIX
BASICS ON ANALYTIC COMBINATORICS OF LATTICE PATHS

We refer mostly to the seminal work of Banderier and Flajolet [2].

Definition 1 (Lattice Path or Walk): A *lattice path* (or walk) is a sequence $(v_1, \ldots, v_n) \in \mathcal{S}^n$ where n is the *length* of the path and

$$\mathcal{S} = \{(a_1, b_1), \ldots, (a_m, b_m) : (a_i, b_i) \in \mathbb{Z} \times \mathbb{Z}\}$$

is the set of *steps*. A path is:

- *directed* if $a_i > 0$;
- *simple* if $a_i = 1$ (in this case the set of steps is written as $\mathcal{S} = \{b_1, \ldots, b_m\}$);
- *unconstrained* (resp. *constrained*) if $v \in \mathbb{Z} \times \mathbb{Z}$ (resp. $v \in \mathbb{Z} \times \mathbb{Z}_{\geq 0}$).

We can assign a *weight* to each allowed step, that is $\mathcal{S} \ni s \mapsto w(s) \in \mathbb{R}$. The following definition is the starting point of the analytic approach:

Definition 2 (Characteristic Polynomial): Let $\mathcal{S}$ be the set of steps of a simple walk and w_i the weight associated to b_i. The characteristic polynomial of $\mathcal{S}$ is:

$$P(y) = \sum_{i=1}^{m} w_i y^{b_i}.$$

The ending point of a walk is $\sum_{i=1}^{n} v_i$ that, for simple walks, assumes the form $(K - 1, h)$, where h is called *final altitude*. Denote by $\mathcal{W}_{nk}$ the class of walks with length n and final altitude k, and let $W_{nk} = |\mathcal{W}_{nk}|$.

Definition 3 (Generating Function): The *generating function* of $\mathcal{W}_{nk}$ is defined as:

$$W(x, y) = \sum_{n,k} W_{nk} x^n y^k,$$

where $x \in \mathbb{C}$ is a mark for the length and $y \in \mathbb{C}$ is a mark for the final altitude.

The following theorem links $W(x, y)$ with $P(y)$:

Theorem 2: The generating function of a simple walk is:

$$W(x, y) = \frac{1}{1 - xP(y)}. \tag{8}$$

Proof (Sketch): Rewrite $W(x, y)$ as follows:

$$W(x, y) = \sum_n \left[\sum_k W_{nk} y^k \right] x^n = \sum_n w_n(y) x^n,$$

where $w_n(y) = [x^n] W(x, y)$ is a Laurent polynomial in y where $[y^h] w_n(y)$ is the (possibly weighted) number of ways to reach the final altitude h in n steps. Since the only altitude reachable in 0 steps is 0, then $w_0(y) = 1$; at step 1, the reachable altitudes are described by $P(y)$. In general $w_k(y) = w_{k-1}(y) P(y)$, and therefore a summation over $k \geq 0$ yields to eq. (8).

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