Optimum IR-UWB Coding Under Power Spectral Constraints

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***Abstract*—Optimum coded Impulse-Radio Ultra-Wideband (IR-UWB) waveforms under power spectral constraints are investigated. Transmission over a SISO frequency-selective channel is assumed. An analytical solution for the optimum precoder is proposed, constraining compliance with a uniform mask in the frequency-domain and maximum transmitted power. Comparison against numerical solutions for non-uniform masks is also performed and a generalization of the analytical solution is proposed. Finally, practical design solutions and issues are presented.**

I. Introduction

The study of optimum precoding techniques in IR-UWB, DS-CDMA and MIMO systems is a well investigated topic [1]. This work addresses single-user IR-UWB system with power constraints. Constraints on power spectra on transmitted signals mimicking regulatory mask are also taken into account.

The paper is organized as follows: Section II introduces the system model and defines the problem; Section III contains the proposed analytical solution and the numerical results; Section IV presents design issues of practical precoders and Section V contains concluding remarks.

II. System model

Binary communication over a SISO frequency-selective channel, and in particular Impulse-Radio Ultra-Wideband (IR-UWB) with antipodal modulation is considered. Such a system typically fits requirements of low-rate, unstructured underlay networks. In order to provide spectrum shaping, Time-Hopping (TH) is usually adopted [5]; the symbol period T is therefore divided into N chips and each user transmits a unit-energy pulse modulated waveform $\psi(t)$ within a randomly selected chip. In Impulse-Radio, the pulse is much shorter than chip duration, which guarantees the ultra-wideband characteristic. As well known TH also provides the possibility of introducing multiple access by code division [6]–[8], where the number of users is naturally limited by the number of chips N. This work addresses single-user communications. The transmission TH coder plays the sole role of spectrum shaper.

Figure 1 shows the system model under analysis. Note that the system output, z, is a correlation metric based on which SNR must be computed, and in turn the BER. To this end, we may model the TH-IR-UWB system with a discrete I/O relationship (see Fig. 1) of the form

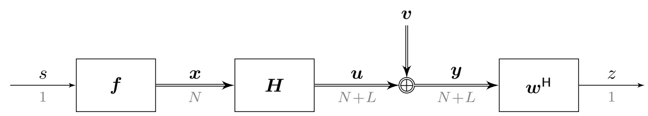


Fig. 1: Discrete-time system model: s is the information-bearing symbol, $\boldsymbol{f} \in \mathbb{R}^N$ the precoder, $\boldsymbol{x} \in \mathbb{R}^N$ the transmitted signal, $\boldsymbol{H} \in \mathbb{R}^{(N+L)\times N}$ the channel matrix, $\boldsymbol{y} = \boldsymbol{u}+\boldsymbol{v} \in \mathbb{R}^{N+L}$ the received signal with desired component $\boldsymbol{u}$ and AWGN $\boldsymbol{v} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I})$, $\boldsymbol{w} \in \mathbb{R}^{(N+L)\times 1}$ the linear receiver and z the correlation metric.

$$\boldsymbol{y}[m] = \boldsymbol{H}_1 \boldsymbol{f}[m]s[m] + \tilde{\boldsymbol{H}}_1 \boldsymbol{f}[m-1]s[m-1] + \boldsymbol{v}[m] \quad (1)$$

where $s[m]$ is the symbol transmitted during the mth symbol period, $\boldsymbol{f}[m] \in \mathbb{R}^N$ is the code vector, $\boldsymbol{H}_1 \in \mathbb{R}^{N\times N}$ is the Toeplitz channel matrix and $\tilde{\boldsymbol{H}}_1 \in \mathbb{R}^{N\times N}$ is the upper triangular matrix accounting IBI. This relationship fits the general framework of block-transmission CDMA systems [2], where each block encodes a symbol. When considering the communication of only one symbol, (1) symplifies into:

$$\boldsymbol{y} = \begin{bmatrix} \boldsymbol{H}_1 \boldsymbol{f}[0] \\ \tilde{\boldsymbol{H}}_1 \boldsymbol{f}[0] \end{bmatrix} s + \begin{bmatrix} \boldsymbol{v}[0] \\ \boldsymbol{v}[1] \end{bmatrix} = \boldsymbol{H}\boldsymbol{f}s + \boldsymbol{v} \quad (2)$$

In (2), $\boldsymbol{y}$ may represent either the discrete-time received waveform or the chip-matched received waveform sampled at chip-rate. In both cases, $\boldsymbol{H}$ is a linear convolution channel matrix, but only in the first case it is straightforwardly derived from the discrete-time channel impulse response (CIR).

The optimum linear receiver is the filter matched to the desired received signal $\boldsymbol{c} := \boldsymbol{H}\boldsymbol{f}$. It is named Rake correlator [4] when the channel is multipath, as is usually the case in UWB and spread-spectrum systems in general. Given the correlation metric:

$$z = \boldsymbol{w}^{\mathsf{H}}\boldsymbol{y} = \boldsymbol{w}^{\mathsf{H}}\boldsymbol{H}\boldsymbol{f}s + \boldsymbol{w}^{\mathsf{H}}\boldsymbol{v},$$

the signal-to-noise ratio, assuming an AWGN vector with identity covariance matrix, $\boldsymbol{v} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I})$, is:

$$\mathrm{SNR} = \frac{\mathbb{E}(\boldsymbol{w}^{\mathsf{H}}\boldsymbol{H}\boldsymbol{f}\,s)^2}{\mathbb{E}(\boldsymbol{w}^{\mathsf{H}}\boldsymbol{v}\boldsymbol{v}^{\mathsf{H}}\boldsymbol{w})} = \frac{(\boldsymbol{w}^{\mathsf{H}}\boldsymbol{H}\boldsymbol{f})^2}{\boldsymbol{w}^{\mathsf{H}}\boldsymbol{w}} = \frac{(\boldsymbol{w}^{\mathsf{H}}\boldsymbol{c})^2}{\boldsymbol{w}^{\mathsf{H}}\boldsymbol{w}},$$

that is maximized in $\boldsymbol{w}$ when $\boldsymbol{w} \propto \boldsymbol{c}$. If $\boldsymbol{w} = \boldsymbol{c}$, SNR reaches its maximum, $\mathrm{SNR}^\star(\boldsymbol{f}) = \boldsymbol{f}^{\mathsf{H}}\boldsymbol{H}^{\mathsf{H}}\boldsymbol{H}\boldsymbol{f}$, and only depends on the energy of the useful received signal.

Assuming perfect channel state information at the transmitter as well as at the receiver, this work aims at finding the optimum $\boldsymbol{f}$, that is, the precoder that maximizes $\mathrm{SNR}^{\star}$, under the following constraints:

(**c1**) transmitted power constraint: the power of transmitted signals is lower than or equal to P;

(**c2**) spectral power constraint: the power spectral density of the transmitted signal has to be compliant with a mask.

The problem can, therefore, be formulated as follows:

$$\begin{cases} \underset{\boldsymbol{f}\in\mathbb{R}^N}{\text{maximize}} & \boldsymbol{f}^{\mathsf{H}}\boldsymbol{H}^{\mathsf{H}}\boldsymbol{H}\boldsymbol{f} \\ \text{subject to} & \mathsf{Tr}(\boldsymbol{f}\boldsymbol{f}^{\mathsf{H}}) \leq \mathsf{P} \\ & |\boldsymbol{W}\boldsymbol{f}|^2 \leq \boldsymbol{\mu} \end{cases}, \tag{3}$$

where $\boldsymbol{W} \in \mathbb{R}^{N\times N}$ is the Fourier matrix,

$$[\boldsymbol{W}]_{mn} := e^{j2\pi mn/N}$$

and $\boldsymbol{\mu}$ represents the mask.

III. Analysis and Results

Observe that, neither $\boldsymbol{H}_1$ nor $\boldsymbol{H}$ is circulant but can both be diagonalized by a Fourier matrix when $N \to \infty$. Using this approximation, problem (3) can be reformulated by discarding $\tilde{\boldsymbol{H}}_1$ and considering $\boldsymbol{H}_1$ as circulant. With an abuse of notation, we call this circulant matrix $\boldsymbol{H}_1$ as $\boldsymbol{H}$:

$$\begin{cases} \underset{\tilde{\boldsymbol{f}}\in\mathbb{C}^N}{\text{maximize}} & \tilde{\boldsymbol{f}}^{\mathsf{H}}\boldsymbol{\Lambda}\tilde{\boldsymbol{f}} \\ \text{subject to} & \mathsf{Tr}(\tilde{\boldsymbol{f}}\tilde{\boldsymbol{f}}^{\mathsf{H}}) \leq \mathsf{P} \\ & |\tilde{\boldsymbol{f}}|^2 \leq \boldsymbol{\mu} \end{cases}, \tag{4}$$

where $\tilde{\boldsymbol{f}} := \boldsymbol{W}\boldsymbol{f}$ and $\boldsymbol{\Lambda}$ is diagonal with $[\boldsymbol{\Lambda}]_{ii}^2 := \lambda_i = |H[i]|^2$ being $H[i]$ the DFT of the first column of $\boldsymbol{H}$.

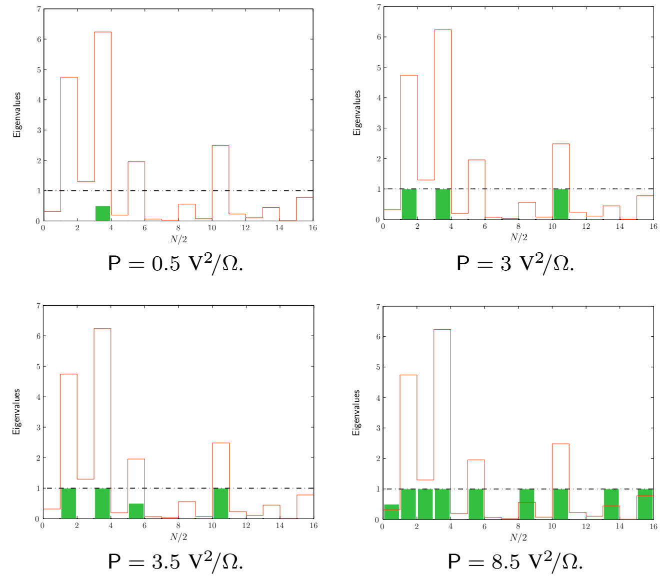
The above approximation is adopted rather than matrix decomposition since it allows to obtain a simple interpretation of results as will shown below.

For the sake of simplicity, we address the problem with a constant mask, that is $\boldsymbol{\mu} := \mu\mathbf{1}$. Let us define $\boldsymbol{\phi} := |\tilde{\boldsymbol{f}}|$. The previous problem is equivalent to:

$$\begin{cases} \underset{\boldsymbol{\phi}\in\mathbb{R}^N}{\text{maximize}} & \boldsymbol{\phi}^{\mathsf{T}}\boldsymbol{\Lambda}\boldsymbol{\phi} \\ \text{subject to} & \mathsf{Tr}(\boldsymbol{\phi}\boldsymbol{\phi}^{\mathsf{T}}) \leq \mathsf{P} \\ & \boldsymbol{\phi} \leq \sqrt{\mu}\mathbf{1} \\ & \boldsymbol{\phi} \geq 0 \end{cases}. \tag{5}$$

This problem is a quadratic programming problem that can be transformed into the following linear programming problem, solvable with well known algorithms (e.g. Dantzig [3]):

$$\begin{cases} \underset{\boldsymbol{\xi}\in\mathbb{R}^N}{\text{maximize}} & \boldsymbol{d}(\boldsymbol{\Lambda})^{\mathsf{T}}\boldsymbol{\xi} \\ \text{subject to} & \begin{bmatrix}\mathbf{1}^{\mathsf{T}} \\ \boldsymbol{I}\end{bmatrix}\boldsymbol{\xi} \leq \begin{bmatrix}1 \\ \boldsymbol{\mu}\end{bmatrix} \end{cases} \tag{6}$$



$\mathsf{P} = 0.5\ \mathsf{V}^2/\Omega$. $\mathsf{P} = 3\ \mathsf{V}^2/\Omega$. $\mathsf{P} = 3.5\ \mathsf{V}^2/\Omega$. $\mathsf{P} = 8.5\ \mathsf{V}^2/\Omega$.

FIG. 2: Example of allocation (filled regions) with constant mask (dashed line) $\mu = 1$, given channel eigenvalues (solid line), for four values of transmitted power $\mathsf{P} = \{0.5, 3, 3.5, 8.5\}$ V^2/Ω.

where $\boldsymbol{d}(\boldsymbol{\Lambda})$ represents the diagonal of $\boldsymbol{\Lambda}$ and $\xi_i := \phi_i^2 = |\tilde{f}_i|^2$.

The peculiar form of the set of constraints suggests an explicit solution. Let us propose this solution rearranging w.l.o.g. the elements of $\boldsymbol{\Lambda}$ in decreasing order. Sorting correspondingly the elements of $\boldsymbol{\phi}$, we have:

$$\phi_\ell^{\star} = \min\{\mu, \max\{0, \mathsf{P} - (\ell-1)\mu\}\},\ \ \ell = 1, \dots, N.$$

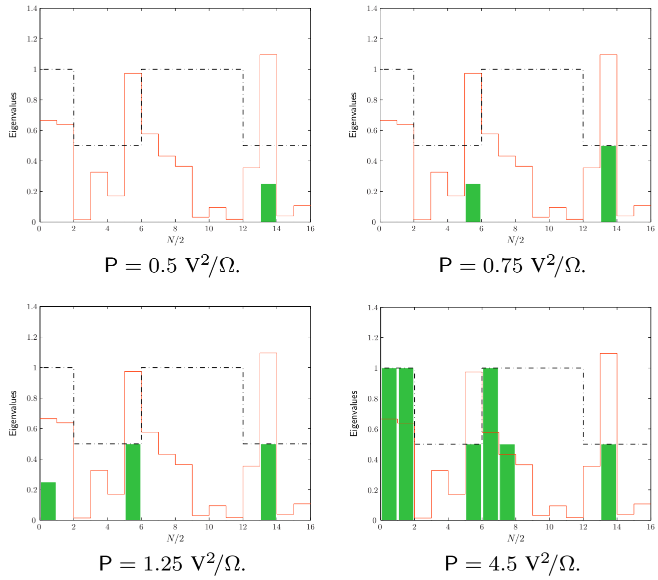
This solution states that the power is allocated increasingly, up to the mask μ, at the eigenmodes (frequencies) associated with highest eigenvalues (channel gains). Therefore, the optimum precoder excites those eigenmodes with a proper power.

An equivalent but simpler way to write this solution is the following: suppose that the total power can be written as $\mathsf{P} = k\mu + r$, $0 \leq r < \mu$, with $k < N$. Then:

$$\begin{aligned} \phi_1^{\star} &= \dots = \phi_k^{\star} = \mu, \\ \phi_{k+1}^{\star} &= r, \\ \phi_{k+2}^{\star} &= \dots = \phi_N^{\star} = 0. \end{aligned}$$

A comparison with a numerical solution is shown in FIG. 2 for the case of uniform mask. For a non-uniform mask, as suggested by FIG. 3, the idea behind the analytical solution remains the same.

If $k = 0$, the constraint on the mask is relaxed and all the power is allocated to the highest eigenvalue. This could be the case, once having fixed the mask, when transmitted power is very low. The precoder $\boldsymbol{f}$ is given by the eigenvector of $\boldsymbol{H}^{\mathsf{H}}\boldsymbol{H}$ corresponding to the maximum eigenvalue $\lambda_{\max}$, or equivalently by the right singular vector of $\boldsymbol{H}$ corresponding to the maximum singular value $\sigma_{\max}$. It turns out that $\lambda_{\max}(\boldsymbol{H}^{\mathsf{H}}\boldsymbol{H}) = \sigma_{\max}^2(\boldsymbol{H})$.



$\mathsf{P} = 0.5\ \mathrm{V}^2/\Omega$. $\mathsf{P} = 0.75\ \mathrm{V}^2/\Omega$.

$\mathsf{P} = 1.25\ \mathrm{V}^2/\Omega$. $\mathsf{P} = 4.5\ \mathrm{V}^2/\Omega$.

FIG. 3: Example of allocation (filled regions) with non-uniform mask (dashed line) given channel eigenvalues (solid line), for four values of transmitted power $\mathsf{P} = \{0.5, 0.75, 1.25, 4.5\}\ \mathrm{V}^2/\Omega$.

If $k \geq N$, the power is distributed in order to fit the mask, therefore using all the frequencies.

Moreover, a potential bandwidth constraint can be translated into a constraint on the usage of contiguous eigenvalues. The set of eigenvalues can be partitioned, in general, in $N-K+1$ subsets, each with K contiguous elements, and the previous problem can be solved in the subspace spanned by the eigenvectors associated with those eigenvalues. The optimum subset depends in general on the mask and the transmitted power P. However, when the transmitted power is sufficiently high, e.g. $\mathsf{P} \gg \max(\boldsymbol{\mu})$, the optimum set is the one with the higher sum of eigenvalues.

IV. Design issues

This section aims at proposing some insight for practical design of the transmitted waveform once known the optimum $\boldsymbol{f}^\star$. Let us split the analysis according to the meaning of (1) and (2).

A. Sampled waveform

In this case $\boldsymbol{y}[m]$ contains the samples of the received waveform at a sufficient rate, say $1/\Delta t$, according to the bandwidth of ψ. Then $\boldsymbol{f}$ contains the samples of the transmitted waveform. Therefore, finding the optimum $\boldsymbol{f}$, we found directly the transmitter impulse response. In this case, we stress that the firsts $L+1$ entries of the first column of $\boldsymbol{H}$ represent the discrete-time channel impulse response whose order is L and duration is $L\Delta t$. Moreover, in this case we have to be able to transmit a general waveform, even non impulsive but continuous, that depends on the channel and match the received desired signal at the receiver.

B. Chip-matched sampled waveform

It is a different case that of $\boldsymbol{y}[m]$ containing the samples of the chip-matched received waveform taken at least at chip-rate. In this case the shaper is fixed: its output is $\psi(t)$. The nth entry of $\boldsymbol{f}$ represents the amplitude at which a pulse is transmitted on the nth chip. The transmitted waveform is (let be T_c the chip and T the symbol duration)

$$x(t) = \sum_{n=0}^{N-1} f_n \psi(t - nT_c), \quad t \in [0, NT_c).$$

The matrix $\boldsymbol{H}$ summarizes the effect of the channel as seen at the receiver after the chip-matching of the received waveform: it is no longer simply the discrete-time channel impulse response. Moreover, the power constraint and the mask will have the same form as in the previous optimization problem, but in general it assumes a different value. The amplitude spectrum of the transmitted signal is

$$|X(\nu)|^2 = |F(e^{j\omega(\nu)})|^2 |\Psi(\nu)|^2,$$

where $F(e^{j\omega})$ is the DTFT of the sequence $(f_n)_{n=0}^{N-1}$ and ω is a function of ν, namely $\omega = 2\pi\nu T_c$. This spectrum, well known in PAM modulation, is composed of a periodic term with period $1/T_c$ modulated by the pulse spectrum. Moreover, recognise that it is directly linked to the optimum vector $\boldsymbol{\phi}^\star$ found as the solution of the optimization problem (5), namely $|X(n/T)|^2 = |F(e^{j2\pi n/N})|^2 |\Psi(n/T)|^2 = \phi_n^2 |\Psi(n/T)|^2$.

V. Concluding remarks

In this paper we analysed the optimum precoding of a single-user IR-UWB communication over a frequency-selective channel under spectral constraints on the transmitted waveform. The problem is reduced to the optimum beamforming of the sequence of transmitted pulses over N consecutive chips. The optimum precoder $\boldsymbol{f}$ acts like a beamformer that excites the eigenmodes of the channel associated with highest eigenvalues with a power that depends on the constraints.

In the case of constant mask, when the transmitted power is high, e.g. $\mathsf{P} \gg \max(\boldsymbol{\mu})$, the optimum allocation is uniform and the spectrum of the transmitted signal is nearly flat. This can be achieved by a totally uncorrelated sequence (e.g. a pseudo-random sequence spanning all chips), or by an impulsive sequence (e.g. a time-hopping, where only one chip is used for transmission) provided that the pulse bandwidth is sufficiently large. In both cases, no knowledge about the channel is required since power is spread over all frequencies, as in a spread-spectrum system.

On the other hand, when power is allocated on one eigenmode only, then $\boldsymbol{f}$ has to be stronlgy correlated: for instance, it can be approximately sinusoidal (notice that remains a freedom in the phase spectrum of $\tilde{\boldsymbol{f}}$ that can be exploited). It is the envelope of $x(t)$ that models the spectrum. Note however that this case corresponds to low P values: although power is concentrated on one eigenvalue, the power spectral density do not violate the mask.

Finally, intermediate choices of power allocation may be viewed as the superimposition of different envelopes modulating the pulse train.

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