Communications Theory and Engineering

Master's Degree in Electronic Engineering

Sapienza University of Rome

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Block codes and convolutional codes

Parity check matrix

Soft and hard decoders find codewords closest (in Euclidean or Hamming distances) to the received block.

Implementation becomes difficult for large and k and n, since there are 2^k distances that need to be computed and compared.

For hard decoding it is possible to implement efficient decoding techniques

Parity check matrix

Consider a systematic linear *(n, k)* block code, which has a generator matrix of the form

$$G = \left[I_k \mid P \right]$$

Given a row vector b of k bits, the corresponding codeword is $c=bG$, a row vector which can be written

$$c = \left[b \quad a \right]$$

where $a=bP$ is a row vector with n-k parity-check bits and one has

$$bP \oplus a = 0$$

$$\left[b \quad a \right] \begin{bmatrix} P \\ I_{n-k} \end{bmatrix} = 0$$

Parity check matrix

$$bP \oplus a = 0$$

$$\begin{bmatrix} b & a \end{bmatrix} \begin{bmatrix} P \\ I_{n-k} \end{bmatrix} = 0$$

or

$$cH' = 0 \quad with$$

$$H = \begin{bmatrix} P' | I_{n-k} \end{bmatrix}$$

H is called the parity-check matrix

H can be used to check if a vector c is a codeword

Example: the code for parity control *(k + 1, k)*

For the parity check code *(k + 1, k)* the parity check matrix is

$$H = \begin{bmatrix} 1 & 1 & 1 & . & . & . & 1 & 1 \end{bmatrix}$$

One can check in fact if a bit vector is a codeword by summing (modulo-two) all of the bits and checking if the sum is zero.

Example - Hamming code (7,4)

The parity-check matrix for the *(7,4)* Hamming code is

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

c_2 =[0110101], is not a codeword and c_2H'= [100]

while c = [0110001] is a code word since cH'=0

Syndrome

A parity check matrix can be found for all linear codes even though they are non-systematic block

The parity check matrix is a compact representation of a code

For a received vector c_r a transmitted codeword c_t and the error pattern e, one has:

$$c_r = c_t \oplus e$$

And the syndrome is defined

$$s = c_r H' = c_t H' + eH' = eH'$$

The syndrome is zero if c_r is a codeword, which occurs if and only if e is a codeword (recall that 0 is always a codeword of a linear code). Efficient decoders use the syndrome to flag the position of an error, which can then be corrected

Hamming codes

For any positive integer m there exists a Hamming code with

$$(n,k) = (2^m - 1, 2^m - 1 - m)$$

The parity-check matrix has n columns each with n–k bits

For a Hamming code, the parity-check matrix is constructed by letting the $n=2^m-1$ columns be all possible binary vectors with m=n-k elements, except the zero vector

Cyclic codes

The most practical block codes are cyclic codes

An (n,k) linear block code is said to be cyclic if any cyclic shift of a codeword produces another codeword

Some Hamming codes are cyclic codes

The algebraic properties of cyclic codes permit collapsing the information contained by the generator matrix into a single polynomial, called the generator polynomial, or by the dual parity polynomial

BCH and Reed-Solomon codes

The most important cyclic codes are the BCH codes

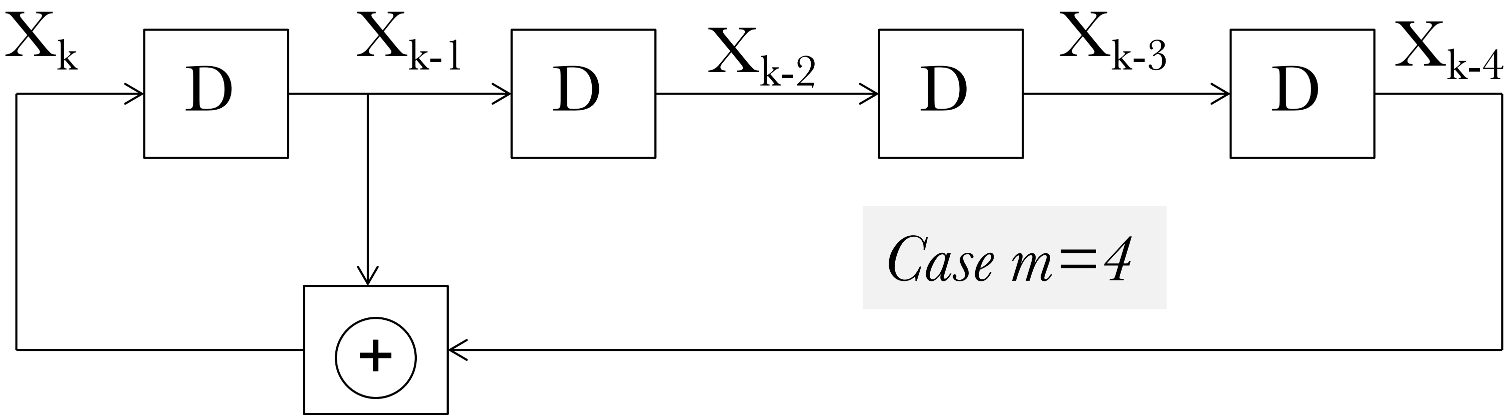
Bose, Ray-Chaudhuri, Hocquenghem, 1960

These are very efficient codes that allow the correction of multiple errors (satellite communications, CDs, DVDs)

A subset of these codes are the very famous Reed-Solomon codes (Voyager, CD, Blueray, QRcodes, Wimax, DSL, DVB)

Maximal-Length Shift Register Codes

Maximal-length shift register codes



The shift register is loaded with *4* bits and clocked 2^m*-1* times

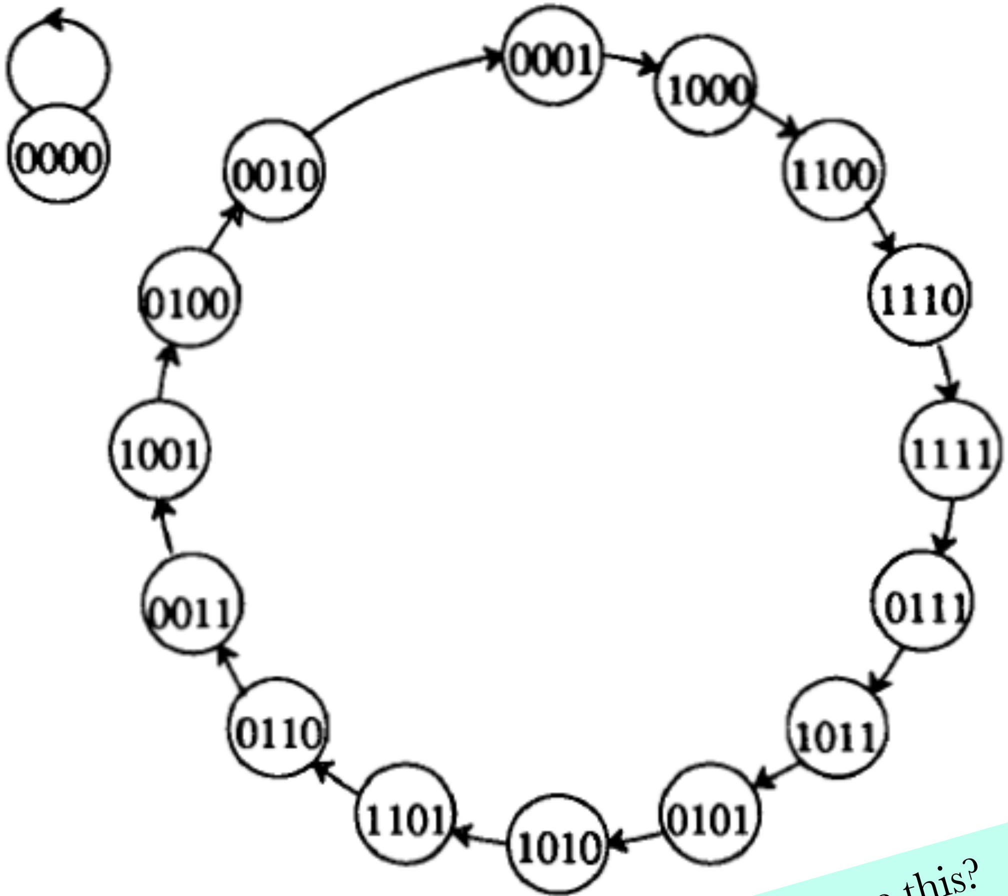
The output forms the 2^m*-1* length code

The result is an $(n,k)=(2^m-1, m)$ block code

The output is periodic with period 2^m-1

The code is cyclical

Example: state transition matrix for the m=4 maximal-length shift register



This is a cyclic code *(15,4)* in the case *m=4*

$$t = \left\lfloor \frac{d_{H,\min} - 1}{2} \right\rfloor$$

The Hamming weight is 2^{m-1}

Can you prove this?

The minimum Hamming distance is *8*

Correction capacity: *3* bit errors

Convolutional codes

A convolutional coder is a finite memory system (rather than a memoryless system, as in the case of the block coder).

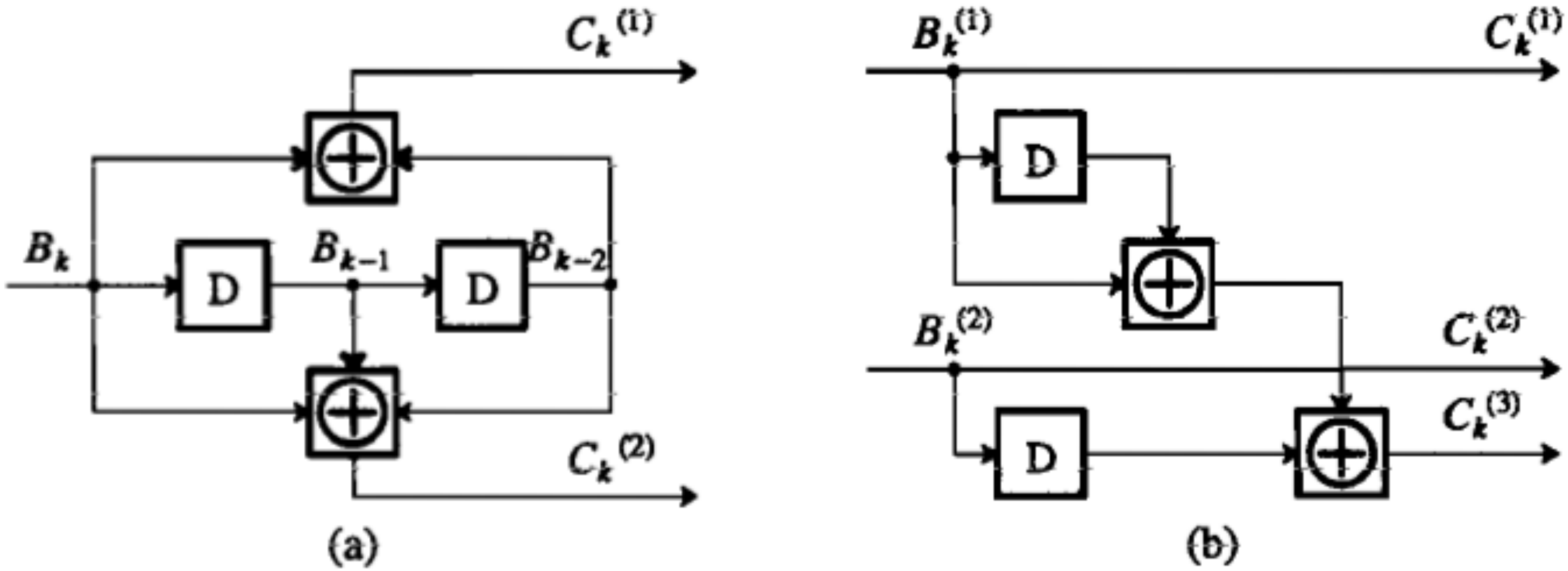
The name refers to the fact that the added redundant bits are generated by modulo-two convolutions

Convolutional codes are widely used because they provide better performance than block codes

A good part of their performance is attributable to the availability of practical soft decoding techniques.

As for block codes, linear convolutional coders are constructed using modulo-two adders with the addition of delay elements.

Example of convolutional codes



Conv*(1/2)*
Rate *1/2*

Conv*(2/3)*
Rate *2/3*
Note: it is systematic

Generator matrix

Example of the conv*(1/2)* case

$$G(D) = \left[1 \oplus D^2, 1 \oplus D \oplus D^2\right]$$

Example of the conv*(2/3)* case

$$G(D) = \begin{bmatrix} 1 & 0 & 1 \oplus D \\ 0 & 1 & D \end{bmatrix}$$

Parity check matrix

Example
conv*(2/3)*

$$G(D)=\begin{bmatrix} 1 & 0 & 1\oplus D \\ 0 & 1 & D \end{bmatrix}=\left[I_k \mid P\right]$$

$$H=\left[P' \mid I_{n-k}\right]=\left[1\oplus D, D, 1\right]$$

Verify that for all codewords one has

$$C(D)H'(D)=0$$

$$C_k^{(1)}\oplus C_{k-1}^{(1)}\oplus C_{k-1}^{(2)}=C_k^{(3)}$$

$$\Rightarrow C^{(3)}(D)=(1\oplus D)C^{(1)}(D)\oplus DC^{(2)}(D)$$

$$\Rightarrow (1\oplus D)C^{(1)}(D)\oplus DC^{(2)}(D)\oplus C^{(3)}(D)=0$$

$$\Rightarrow C(D)H'(D)=0$$

Length of a convolutional code

The constraint length of a convolutional code is defined as *1+* the maximum degree of the polynomials in the generator matrix

$$M = 1 + \max_{i,j}\left[\deg\left(g_{ij}(D)\right)\right]$$

For conv*(1/2)*, *M=3*

For conv*(2/3)*, *M=2*